Chapter Three

Algebraic Expressions

Algebraic formulae are used to solve many algebraic problems. Moreover, many algebraic expressions are presented by resolving them into factors. That is why the problem solved by algebraic formulae and the contents of resolving expressions into factors by making suitable for the students have been presented in this chapter. Moreover, different types of mathematical problems can be solved by resolving into factors with the help of algebraic formulae. In the previous class, algebraic formulae and their related corollaries have been discussed elaborately. In this chapter, those are reiterated and some of their applications are presented through examples. Besides, extension of the formulae of square and cube, resolution into factors using remainder theorem and formation of algebraic formulae and their applications in solving practical problems have been discussed here in detail.

At the end of the chapter, the students will be able to –

- Expand the formulae of square and cube by applying algebraic formulae
- Explain the remainder theorem and resolve into factors by applying the theorem
- Form algebraic formulae for solving real life problems and solve the problems by applying the formulae.

3·1 Algebraic Expressions

Meaningful organization of operational signs and numerical letter symbols is called algebraic expression. Such as, $2a+3b-4c$ is an algebraic expression. In algebraic expression, different types of information are expressed through the letters $a, b, c, p, q, r, m, n, x, y, z, ...$ etc. These alphabet are used to solve different types of problems related to algebraic expressions. In arithmetic, only positive numbers are used, where as, in algebra, both positive and negative numbers including zero are used. Algebra is the generalization form of arithmetic. The numbers used in algebraic expressions are constants, their values are fixed.

The letter symbols used in algebraic expressions are variables, their values are not fixed, they can be of any value.

3·2 Algebraic Formulae

Any general rule or resolution expressed by algebraic symbols is called Algebraic Formula. In class VII and VIII, algebraic formulae and related corollaries have been discussed. In this chapter, some applications are presented on the basis of that discussion.

Formula 1. $(a+b)^2 = a^2 + 2ab + b^2$

Formula 2. $(a-b)^2 = a^2 - 2ab + b^2$

Remark : It is seen from formula 1 and formula 2 that, adding $2ab$ or $-2ab$ with a^2+b^2, we get a perfect square, i.e. we get, $(a+b)^2$ or $(a-b)^2$. Substituting $-b$ instead of b in formula 1, we get formula 2 :

$$\{a+(-b)\}^2 = a^2 + 2a(-b) + (-b)^2$$

That is, $(a-b)^2 = a^2 - 2ab + b^2$.

Corollary 1. $a^2 + b^2 = (a+b)^2 - 2ab$

Corollary 2. $a^2 + b^2 = (a-b)^2 + 2ab$

Corollary 3. $(a+b)^2 = (a-b)^2 + 4ab$

Proof :
$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ &= a^2 - 2ab + b^2 + 4ab\\ &= (a-b)^2 + 4ab\end{aligned}$$

Corollary 4. $(a-b)^2 = (a+b)^2 - 4ab$

Proof :
$$\begin{aligned}(a-b)^2 &= a^2 - 2ab + b^2\\ &= a^2 + 2ab + b^2 - 4ab\\ &= (a+b)^2 - 4ab\end{aligned}$$

Corollary 5. $a^2 + b^2 = \dfrac{(a+b)^2 + (a-b)^2}{2}$

Proof : From formula 1 and formula 2,

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

Adding, $2a^2 + 2b^2 = (a+b)^2 + (a-b)^2$

or, $2(a^2+b^2) = (a+b)^2 + (a-b)^2$

Hence, $(a^2+b^2) = \dfrac{(a+b)^2 + (a-b)^2}{2}$

Corollary 6. $ab = \left(\dfrac{a+b}{2}\right)^2 - \left(\dfrac{a-b}{2}\right)^2$

Proof : From formula 1 and formula 2,

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

Subtracting, $4ab = (a+b)^2 - (a-b)^2$

or, $ab = \frac{(a+b)^2}{4} - \frac{(a-b)^2}{4}$

Hence, $ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$

Remark : Product of any two quantities can be expressed as the difference of two squares by applying the corollary 6.

Formula 3. $a^2 - b^2 = (a+b)(a-b)$

That is, the difference of the squares of two expressions = sum of two expressions × difference of two expressions.

Formula 4. $(x+a)(x+b) = x^2 + (a+b)\,x + ab$

That is, $(x+a)(x+b) = x^2 +$ (algebraic sum of a and b) $x +$ (the product of a and b)

Extension of Formula for Square

There are three terms in the expression $a+b+c$. It can be considered the sum of two terms $(a+b)$ and c.

Therefore, by applying formula 1, the square of the expression $a+b+c$ is,

$(a+b+c)^2 = \{(a+b)+c\}^2$

$= (a+b)^2 + 2(a+b)c + c^2$

$= a^2 + 2ab + b^2 + 2ac + 2bc + c^2$

$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$.

Formula 5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$.

Corollary 7. $a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac)$

Corollary 8. $2(ab+bc+ac) = (a+b+c)^2 - (a^2+b^2+c^2)$

Observe : Applying formula 5, we get,

(i) $(a+b-c)^2 = \{a+b+(-c)\}^2$

$= a^2 + b^2 + (-c)^2 + 2ab + 2b(-c) + 2a(-c)$

$= a^2 + b^2 + c^2 + 2ab - 2bc - 2ac$

(ii) $(a-b+c)^2 = \{a+(-b)+c\}^2$

$= a^2 + (-b)^2 + c^2 + 2a(-b) + 2(-b)c + 2ac$

$= a^2 + b^2 + c^2 - 2ab - 2bc + 2ac$

(iii) $(a-b-c)^2 = \{a+(-b)+(-c)\}^2$

$= a^2 + (-b)^2 + (-c)^2 + 2a(-b) + 2(-b)(-c) + 2a(-c)$

$= a^2 + b^2 + c^2 - 2ab + 2bc - 2ac$

Example 1. What is the square of $(4x + 5y)$?

Solution : $(4x + 5y)^2 = (4x)^2 + 2 \times (4x) \times (5y) + (5y)^2$

$= 16x^2 + 40xy + 25y^2$

Example 2. What is the square of $(3a - 7b)$?

Solution : $(3a - 7b)^2 = (3a)^2 - 2 \times (3a) \times (7b) + (7b)^2$

$= 9a^2 - 42ab + 49b^2$

Example 3. Find the square of 996 by applying the formula of square.

Solution : $(996)^2 = (1000 - 4)^2$

$= (1000)^2 - 2 \times 1000 \times 4 + (4)^2$

$= 1000000 - 8000 + 16$

$= 1000016 - 8000$

$= 992016$

Example 4. What is the square of $a + b + c + d$?

Solution : $(a + b + c + d)^2 = \{(a + b) + (c + d)\}^2$

$= (a + b)^2 + 2(a + b)(c + d) + (c + d)^2$

$= a^2 + 2ab + b^2 + 2(ac + ad + bc + bd) + c^2 + 2cd + d^2$

$= a^2 + 2ab + b^2 + 2ac + 2ad + 2bc + 2bd + c^2 + 2cd + d^2$

$= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$

Activity : Find the square with the help of the formulae :
1. $3xy + 2ax$ 2. $4x - 3y$ 3. $x - 5y + 2z$

Example 5. Simplify :

$(5x + 7y + 3z)^2 + 2(7x - 7y - 3z)(5x + 7y + 3z) + (7x - 7y - 3z)^2$

Solution : Let, $5x + 7y + 3z = a$ and $7x - 7y - 3z = b$

$\therefore$ Given expression = $a^2 + 2.b.a + b^2$

$= a^2 + 2ab + b^2$

$= (a + b)^2$

$= \{(5x + 7y + 3z) + (7x - 7y - 3z)\}^2$

[substituting the values of a and b]

$= (5x + 7y + 3z + 7x - 7y - 3z)^2$

$= (12x)^2$

$= 144x^2$

Example 6. If $x-y=2$ and $xy=24$, what is the value of $x+y$?

Solution : $(x+y)^2=(x-y)^2+4xy=(2)^2+4\times 24=4+96=100$

$\therefore\ x+y=\pm\sqrt{100}=\pm 10$

Example 7. If $a^4+a^2b^2+b^4=3$ and $a^2+ab+b^2=3$, what is the value of a^2+b^2?

Solution : $a^4+a^2b^2+b^4=(a^2)^2+2a^2b^2+(b^2)^2-a^2b^2$

$=(a^2+b^2)^2-(ab)^2$

$=(a^2+b^2+ab)(a^2+b^2-ab)$

$=(a^2+ab+b^2)(a^2-ab+b^2)$

$\therefore\ 3=3(a^2-ab+b^2)$ [substituting the values]

or, $a^2-ab+b^2=\frac{3}{3}=1$

Now adding, $a^2+ab+b^2=3$ and $a^2-ab+b^2=1$ we get, $2(a^2+b^2)=4$

or, $a^2+b^2=\frac{4}{2}=2$

$\therefore\ a^2+b^2=2$

Example 8. Prove that, $(a+b)^4-(a-b)^4=8ab(a^2+b^2)$

Solution : $(a+b)^4-(a-b)^4=\{(a+b)^2\}^2-\{(a-b)^2\}^2$

$=\{(a+b)^2+(a-b)^2\}\{(a+b)^2-(a-b)^2\}$

$=2(a^2+b^2)\times 4ab$

$[\because (a+b)^2+(a-b)^2=2(a^2+b^2)$ and $(a+b)^2-(a-b)^2=4ab]$

$=8ab(a^2+b^2)$

$\therefore\ (a+b)^4-(a-b)^4=8ab(a^2+b^2)$

Example 9. If $a+b+c=15$ and $a^2+b^2+c^2=83$, what is the value of $ab+bc+ac$?

Soluton :

Here, $2(ab+bc+ac)$

$=(a+b+c)^2-(a^2+b^2+c^2)$

$=(15)^2-83$

$=225-83$

$=142$

$\therefore\ ab+bc+ac=\frac{142}{2}=71$

Alternative method,

We know ,

$(a+b+c)^2=(a^2+b^2+c^2)+2(ab+bc+ac)$

or, $(15)^2=83+2(ab+bc+ac)$

or, $225-83=2(ab+bc+ac)$

or, $2(ab+bc+ac)=142$

$\therefore\ ab+bc+ac=\frac{142}{2}=71$

Example 10. If $a+b+c=2$ and $ab+bc+ac=1$, what is the value of $(a+b)^2+(b+c)^2+(c+a)^2$?

Solution : $(a+b)^2+(b+c)^2+(c+a)^2$

$= a^2+2ab+b^2+b^2+2bc+c^2+c^2+2ca+a^2$

$= (a^2+b^2+c^2+2ab+2bc+2ca)+(a^2+b^2+c^2)$

$= (a+b+c)^2+\{(a+b+c)^2-2(ab+bc+ac)\}$

$= (2)^2+(2)^2-2\times 1$

$= 4+4-2=8-2=6$

Example 11. Express $(2x+3y)(4x-5y)$ as the difference of two squares.

Solution : Let, $2x+3y=a$ and $4x-5y=b$

$\therefore$ Given expression = $ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2$

$=\left(\frac{2x+3y+4x-5y}{2}\right)^2-\left(\frac{2x+3y-4x+5y}{2}\right)^2$ [substituting the values of a and b]

$=\left(\frac{6x-2y}{2}\right)^2-\left(\frac{8y-2x}{2}\right)^2$

$=\left(\frac{2(3x-y)}{2}\right)^2-\left(\frac{2(4y-x)}{2}\right)^2$

$=(3x-y)^2-(4y-x)^2$

$\therefore\ (2x+3y)(4x-5y)=(3x-y)^2-(4y-x)^2$

Activity : 1. Simplify : $(4x+3y)^2+2(4x+3y)(4x-3y)+(4x-3y)^2$
2. If $x+y+z=12$ and $x^2+y^2+z^2=50$, find the value of $(x-y)^2+(y-z)^2+(z-x)^2$.

Exercise 3·1

1. Find the square with the help of the formulae :

(a) $2a+3b$ (b) $2ab+3bc$ (c) $x^2+\frac{2}{y^2}$ (d) $a+\frac{1}{a}$ (e) $4y-5x$ (f) $ab-c$

(g) $5x^2-y$ (h) $x+2y+4z$ (i) $3p+4q-5r$ (j) $3b-5c-2a$ (k) $ax-by-cz$

(l) $a-b+c-d$ (m) $2a+3x-2y-5z$ (n) 101 (o) 997 (p) 1007

2. Simplify :

(a) $(2a+7)^2+2(2a+7)(2a-7)+(2a-7)^2$

(b) $(3x+2y)^2+2(3x+2y)(3x-2y)+(3x-2y)^2$

(c) $(7p+3r-5x)^2-2(7p+3r-5x)(8p-4r-5x)+(8p-4r-5x)^2$

(d) $(2m+3n-p)^2+(2m-3n+p)^2-2(2m+3n-p)(2m-3n+p)$

(e) $6\cdot35\times6\cdot35+2\times6\cdot35\times3\cdot65+3\cdot65\times3\cdot65$

(f) $5874\times5874+3774\times3774-7548\times5874$

(g) $\dfrac{7529\times7529-7519\times7519}{7529+7519}$

(h) $\dfrac{2345\times2345-759\times759}{2345-759}$

3. If $a-b=4$ and $ab=60$, what is the value of $a+b$?
4. If $a+b=7$ and $ab=12$, what is the value of $a-b$?
5. If $a+b=9m$ and $ab=18m^2$, what is the value of $a-b$?
6. If $x-y=2$ and $xy=63$, what is the value of x^2+y^2 ?
7. If $x-\dfrac{1}{x}=4$, prove that, $x^4+\dfrac{1}{x^4}=322$.
8. If $2x+\dfrac{2}{x}=3$, what is the value of $x^2+\dfrac{1}{x^2}$?
9. If $a+\dfrac{1}{a}=2$, show that, $a^2+\dfrac{1}{a^2}=a^4+\dfrac{1}{a^4}$.
10. If $a+b=\sqrt{7}$ and $a-b=\sqrt{5}$, prove that, $8ab(a^2+b^2)=24$.
11. If $a+b+c=9$ and $ab+bc+ca=31$, find the value of $a^2+b^2+c^2$.
12. If $a^2+b^2+c^2=9$ and $ab+bc+ca=8$, what is the value of $(a+b+c)^2$?
13. If $a+b+c=6$ and $a^2+b^2+c^2=14$, find the value of $(a-b)^2+(b-c)^2+(c-a)^2$.
14. If $x+y+z=10$ and $xy+yz+zx=31$, what is the value of $(x+y)^2+(y+z)^2+(z+x)^2$?
15. If $x=3, y=4$ and $z=5$ find the value of $9x^2+16y^2+4z^2-24xy-16yz+12zx$.
16. Prove that, $\left\{\left(\dfrac{x+y}{2}\right)^2-\left(\dfrac{x-y}{2}\right)^2\right\}^2=\left(\dfrac{x^2+y^2}{2}\right)^2-\left(\dfrac{x^2-y^2}{2}\right)^2$.
17. Express $(a+2b)(3a+2c)$ as the difference of two squares.
18. Express $(x+7)(x-9)$ as the difference of two squares.
19. Express $x^2+10x+24$ as the difference of two squares.

20. If $a^4+a^2b^2+b^4=8$ and $a^2+ab+b^2=4$, find the value of (i) a^2+b^2, (ii) ab.

3·3 Formulae of Cubes

Formula 6. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$$= a^3+b^3+3ab(a+b)$$

Proof : $(a+b)^3=(a+b)(a+b)^2$

$$= (a+b)(a^2+2ab+b^2)$$
$$= a(a^2+2ab+b^2)+b(a^2+2ab+b^2)$$
$$= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3$$
$$= a^3+3a^2b+3ab^2+b^3$$
$$= a^3+b^3+3ab(a+b)$$

Corollary 9. $a^3+b^3=(a+b)^3-3ab(a+b)$

Formula 7. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$$= a^3-b^3-3ab(a-b)$$

Proof : $(a-b)^3=(a-b)(a-b)^2$

$$= (a-b)(a^2-2ab+b^2)$$
$$= a(a^2-2ab+b^2)-b(a^2-2ab+b^2)$$
$$= a^3-2a^2b+ab^2-a^2b+2ab^2-b^3$$
$$= a^3-3a^2b+3ab^2-b^3$$
$$= a^3-b^3-3ab(a-b)$$

Observe : Substituting $-b$ instead of b in formula 6, we get formula 7 :

$$\{a+(-b)\}^3=a^3+(-b)^3+3a(-b)\{a+(-b)\}$$

That is, $(a-b)^3=a^3-b^3-3ab(a-b)$

Corollary 10. $a^3-b^3=(a-b)^3+3ab(a-b)$

Formula 8. $a^3+b^3=(a+b)(a^2-ab+b^2)$

Proof : $a^3+b^3=(a+b)^3-3ab(a+b)$

$$= (a+b)\{(a+b)^2-3ab\}$$
$$= (a+b)(a^2+2ab+b^2-3ab)$$
$$= (a+b)(a^2-ab+b^2)$$

Formula 9. $a^3-b^3=(a-b)(a^2+ab+b^2)$

Proof : $a^3-b^3=(a-b)^3+3ab(a-b)$

$$= (a-b)\{(a-b)^2+3ab\}$$

$$= (a-b)(a^2-2ab+b^2+3ab)$$
$$= (a-b)(a^2+ab+b^2)$$

Example 12. Find the cube of $2x+3y$.

Solution : $(2x+3y)^3 = (2x)^3+3(2x)^2\cdot 3y+3\cdot 2x(3y)^2+(3y)^3$

$$= 8x^3+3\cdot 4x^2\cdot 3y+3\cdot 2x\cdot 9y^2+27y^3$$
$$= 8x^3+36x^2y+54xy^2+27y^3$$

Example 13. Find the cube of $2x-y$.

Solution : $(2x-y)^3 = (2x)^3-3\cdot(2x)^2y+3\cdot 2x\cdot y^2-y^3$

$$= 8x^3-3\cdot 4x^2y+6xy^2-y^3$$
$$= 8x^3-12x^2y+6xy^2-y^3$$

Activity : Find the cube with the help of the formulae.
1. $3x+2y$ 2. $3x-4y$ 3. 397

Example 14. If $x=37$, what is the value of $8x^3+72x^2+216x+216$?

Solution : $8x^3+72x^2+216x+216$

$$= (2x)^3+3\cdot(2x)^2\cdot 6+3\cdot 2x.(6)^2+(6)^3$$
$$= (2x+6)^3$$
$$= (2\times 37+6)^3 \text{ [substituting the values]}$$
$$= (74+6)^3$$
$$= (80)^3$$
$$= 512000$$

Example 15. If $x-y=8$ and $xy=5$, what is the value of $x^3-y^3+8(x+y)^2$?

Solution : $x^3-y^3+8(x+y)^2$

$$= (x-y)^3+3xy(x-y)+8\{(x-y)^2+4xy\}$$
$$= (8)^3+3\times 5\times 8+8(8^2+4\times 5) \quad \text{[substituting the values]}$$
$$= 8^3+15\times 8+8(64+20)$$
$$= 8^3+15\times 8+8\times 84$$
$$= 8(8^2+15+84)$$
$$= 8(64+15+84)$$
$$= 8\times 163$$
$$= 1304$$

Example 16. If $a^2 - \sqrt{3}a + 1 = 0$, what is the value of $a^3 + \frac{1}{a^3}$?

Solution : Given that, $a^2 - \sqrt{3}a + 1 = 0$

or, $a^2 + 1 = \sqrt{3}a$ or, $\frac{a^2+1}{a} = \sqrt{3}$

or, $\frac{a^2}{a} + \frac{1}{a} = \sqrt{3}$ or, $a + \frac{1}{a} = \sqrt{3}$

$\therefore$ Given expression $= a^3 + \frac{1}{a^3}$

$= \left(a + \frac{1}{a}\right)^3 - 3a \cdot \frac{1}{a}\left(a + \frac{1}{a}\right)$

$= \left(\sqrt{3}\right)^3 - 3\left(\sqrt{3}\right)$ $[\because a + \frac{1}{a} = \sqrt{3}]$

$= 3\sqrt{3} - 3\sqrt{3}$

$= 0$

Example 17. Simplify :

$(a-b)(a^2+ab+b^2)+(b-c)(b^2+bc+c^2)+(c-a)(c^2+ca+a^2)$

Solution : $(a-b)(a^2+ab+b^2)+(b-c)(b^2+bc+c^2)+(c-a)(c^2+ca+a^2)$

$= a^3 - b^3 + b^3 - c^3 + c^3 - a^3$

$= 0$

Example 18. If $a = \sqrt{3} + \sqrt{2}$, prove that, $a^3 + \frac{1}{a^3} = 18\sqrt{3}$.

Solution : Given that, $a = \sqrt{3} + \sqrt{2}$

$\therefore \frac{1}{a} = \frac{1}{\sqrt{3}+\sqrt{2}}$

$= \frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)}$ [multiplying numerator and denominator by $\left(\sqrt{3}-\sqrt{2}\right)$]

$= \frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}\right)^2 - \left(\sqrt{2}\right)^2}$ $= \frac{\sqrt{3}-\sqrt{2}}{3-2}$

$= \sqrt{3} - \sqrt{2}$

$\therefore a + \frac{1}{a} = \left(\sqrt{3}+\sqrt{2}\right) + \left(\sqrt{3}-\sqrt{2}\right)$

$= \sqrt{3} + \sqrt{2} + \sqrt{3} - \sqrt{2} = 2\sqrt{3}$

Now, $a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\cdot a\cdot\frac{1}{a}\left(a+\frac{1}{a}\right)$

$= \left(2\sqrt{3}\right)^3-3\left(2\sqrt{3}\right)$ $[\because a+\frac{1}{a}=2\sqrt{3}]$

$= 2^3\cdot\left(\sqrt{3}\right)^3-3\times 2\sqrt{3}$

$= 8\cdot 3\sqrt{3}-6\sqrt{3}$

$= 24\sqrt{3}-6\sqrt{3}$

$= 18\sqrt{3}$ (proved)

Activity : 1. If $x=-2$, what is the value of $27x^3-54x^2+36x-8$?

2. If $a+b=5$ and $ab=6$, find the value of $a^3+b^3+4(a-b)^2$.

3. If $x=\sqrt{5}+\sqrt{3}$, find the value of $x^3+\frac{1}{x^3}$.

Exercise 3·2

1. Find the cube with the help of the formulae :
   (a) $2x+5$ (b) $2x^2+3y^2$ (c) $4a-5x^2$ (d) $7m^2-2n$ (e) 403 (f) 998
   (g) $2a-b-3c$ (h) $2x+3y+z$
2. Simplify :
   (a) $(4a-3b)^3-3(4a-3b)^2(2a-3b)+3(4a-3b)(2a-3b)^2-(2a-3b)^3$
   (b) $(2x+y)^3+3(2x+y)^2(2x-y)+3(2x+y)(2x-y)^2+(2x-y)^3$
   (c) $(7x+3b)^3-(5x+3b)^3-6x(7x+3b)(5x+3b)$
   (d) $(x-15)^3+(16-x)^3+3(x-15)(16-x)$
   (e) $(a+b+c)^3-(a-b-c)^3-6(b+c)\{a^2-(b+c)^2\}$
   (f) $(m+n)^6-(m-n)^6-12mn(m^2-n^2)^2$
   (g) $(x+y)(x^2-xy+y^2)+(y+z)(y^2-yz+z^2)+(z+x)(z^2-zx+x^2)$
   (h) $(2x+3y-4z)^3+(2x-3y+4z)^3+12x\{4x^2-(3y-4z)^2\}$
3. If $a-b=5$ and $ab=36$, what is the value of a^3-b^3 ?
4. If $a^3-b^3=513$ and $a-b=3$, what is the value of ab ?
5. If $x=19$ and $y=-12$, find the value of $8x^3+36x^2y+54xy^2+27y^3$.
6. If $a=15$, what is the value of $8a^3+60a^2+150a+130$?
7. If $a=7$ and $b=-5$, what is the value of

$(3a-5b)^3+(4b-2a)^3+3(a-b)(3a-5b)(4b-2a)$?

8. If $a+b=m, a^2+b^2=n$ and $a^3+b^3=p^3$, show that, $m^3+2p^3=3mn$.
9. If $x+y=1$, show that, $x^3+y^3-xy=(x-y)^2$
10. If $a+b=3$ and $ab=2$, find the value of (a) a^2-ab+b^2 and (b) a^3+b^3.
11. If $a-b=5$ and $ab=36$, find the value of (a) a^2+ab+b^2 and (b) a^3-b^3.
12. If $m+\frac{1}{m}=a$, find the value of $m^3+\frac{1}{m^3}$.
13. If $x-\frac{1}{x}=p$, find the value of $x^3-\frac{1}{x^3}$.
14. If $a-\frac{1}{a}=1$, show that, $a^3-\frac{1}{a^3}=4$.
15. If $a+b+c=0$, show that,

    (a) $a^3+b^3+c^3=3abc$ (b) $\frac{(b+c)^2}{3bc}+\frac{(c+a)^2}{3ca}+\frac{(a+b)^2}{3ab}=1$.
16. If $p-q=r$, show that, $p^3-q^3-r^3=3pqr$
17. If $2x-\frac{2}{x}=3$, show that, $8\left(x^3-\frac{1}{x^3}\right)=63$.
18. If $a=\sqrt{6}+\sqrt{5}$, find the value of $\frac{a^6-1}{a^3}$.
19. If $x^3+\frac{1}{x^3}=18\sqrt{3}$, prove that, $x=\sqrt{3}+\sqrt{2}$.
20. If $a^4-a^2+1=0$, prove that, $a^3+\frac{1}{a^3}=0$.

3·4 Resolution into Factors

If an expression is equal to the product of two or more expressions, each of the latter expressions is called a factor of the former expression.

After finding the possible factors of any algebraic expression and then expressing the expression as the product of these factors are called factorization or resolution into factors.

The algebraic expressions may consist of one or more terms. So, the factors may also contain one or more terms.

Some process of resolving expressions into factors :

(a) If any polynomial expression has common factor in every term, at first they are to be found out. For example,

(*i*) $3a^2b+6ab^2+12a^2b^2=3ab(a+2b+4ab)$

$(ii)\ 2ab(x-y)+2bc(x-y)+3ca(x-y)=(x-y)(2ab+2bc+3ca)$

(b) Expressing an expression in the form of a perfect square ;

Example 1. Resolve into factors : $4x^2+12x+9$.

Solution : $4x^2+12x+9=(2x)^2+2\times 2x\times 3+(3)^2$

$= (2x+3)^2=(2x+3)(2x+3)$

Example 2. Resolve into factors : $9x^2-30xy+25y^2$.

Solution : $9x^2-30xy+25y^2$

$= (3x)^2-2\times 3x\times 5y+(5y)^2$

$= (3x-5y)^2=(3x-5y)(3x-5y)$

(c) Expressing an expression as the difference of two squares and then applying the formula $a^2-b^2=(a+b)(a-b)$:

Example 3. Reslove into factors : $a^2-1+2b-b^2$.

Solution : $a^2-1+2b-b^2=a^2-(b^2-2b+1)$

$= a^2-(b-1)^2=\{a+(b-1)\}\{a-(b-1)\}$

$= (a+b-1)(a-b+1)$

Example 4. Resolve into factors : a^4+64b^4.

Solution : $a^4+64b^4=(a^2)^2+(8b^2)^2$

$= (a^2)^2+2\times a^2\times 8b^2+(8b^2)^2-16a^2b^2$

$= (a^2+8b^2)^2-(4ab)^2$

$= (a^2+8b^2+4ab)(a^2+8b^2-4ab)$

$= (a^2+4ab+8b^2)(a^2-4ab+8b^2)$

Activity : Resolve into factors :

1. $abx^2+acx^3+adx^4$ 2. $xa^2-144xb^2$ 3. $x^2-2xy-4y-4$

(d) Using the formula $x^2+(a+b)x+ab=(x+a)(x+b)$:

Example 5. Resolve into factors : $x^2+12x+35$.

Solution : $x^2+12x+35=x^2+(5+7)x+5\times 7$

$= (x+5)(x+7)$

In this method, a polynomial of the form x^2+px+q can be factorized, if two integers a and b can be found so that, it is $a+b=p$ and $ab=q$. For this, two factors of q with their signs are to be taken whose algebraic sum is p. If $q>0$, a and b will be of same signs and if $q<0$, a and b will be of opposite signs.

Example 6. Resolve into factors : $x^2 - 5x + 6$.

Solution : $x^2 - 5x + 6 = x^2 + (-2-3)x + (-2)(-3)$

$= (x-2)(x-3)$

Example 7. Resolve into factors : $x^2 - 2x - 35$.

Solution : $x^2 - 2x - 35$

$= x^2 + (-7+5)x + (-7)(+5)$

$= (x-7)(x+5)$

Example 8. Resolve into factors : $x^2 + x - 20$.

Solution : $x^2 + x - 20$

$= x^2 + (5-4)x + (5)(-4)$

$= (x+5)(x-4)$

(e) By middle term break-up method of polynomial of the form of $ax^2 + bx + c$:

If $ax^2 + bx + c = (rx+p)(sx+q)$

$ax^2 + bx + c = rsx^2 + (rq+sp)x + pq$

That is, $a = rs, b = rq + sp$ and $c = pq$.

Hence, $ac = rspq = (rq)(sp)$ and $b = rq + sp$

Therefore, to determine factors of the polynomial $ax^2 + bx + c$, ac that is, the product of the coefficient of x^2 and the term free from x are to be expressed into two such factors whose algebraic sum is equal to b, the coefficient of x.

Example 9. Resolve into factors : $12x^2 + 35x + 18$.

Solution : $12x^2 + 35x + 18$

Here, $12 \times 18 = 216 = 27 \times 8$ and $27 + 8 = 35$

$\therefore\ 12x^2 + 35x + 18 = 12x^2 + 27x + 8x + 18$

$= 3x(4x+9) + 2(4x+9)$

$= (4x+9)(3x+2)$

Example 10. Resolve into factors : $3x^2 - x - 14$.

Solution : $3x^2 - x - 14 = 3x^2 - 7x + 6x - 14$

$= x(3x-7) + 2(3x-7)$

$= (3x-7)(x+2)$

Activity : Resolve into factors :

1. $x^2 + x - 56$ 2. $16x^3 - 46x^2 + 15x$ 3. $12x^2 + 17x + 6$

(f) Expressing the expression in the form of perfect cubes :

Example 11. Resolve into factors : $8x^3+36x^2y+54xy^2+27y^3$.

Solution : $8x^3+36x^2y+54xy^2+27y^3$

$= (2x)^3+3\times(2x)^2\times 3y+3\times 2x\times(3y)^2+(3y)^3$

$= (2x+3y)^3 = (2x+3y)(2x+3y)(2x+3y)$

(g) Applying the formulae : $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$:

Example 12. Resolve into factors : (i) $8a^3+27b^3$ (ii) a^6-64

Solution : (i) $8a^3+27b^3=(2a)^3+(3b)^3$

$= (2a+3b)\{(2a)^2-2a\times 3b+(3b)^2\}$

$= (2a+3b)(4a^2-6ab+9b^2)$

(ii) $a^6-64 = (a^2)^3-(4)^3$

$= (a^2-4)\{(a^2)^2+a^2\times 4+(4)^2\}$

$= (a^2-4)(a^4+4a^2+16)$

But, $a^2-4=a^2-2^2=(a+2)(a-2)$

and $a^4+4a^2+16=(a^2)^2+(4)^2+4a^2$

$= (a^2+4)^2-2(a^2)(4)+4a^2$

$= (a^2+4)^2-4a^2$

$= (a^2+4)^2-(2a)^2$

$= (a^2+4+2a)(a^2+4-2a)$

$= (a^2+2a+4)(a^2-2a+4)$

$\therefore$ a^6-64

$=(a+2)(a-2)(a^2+2a+4)(a^2-2a+4)$

Alternative method :

$a^6-64 = (a^3)^2-8^2$

$= (a^3+8)(a^3-8)$

$= (a^3+2^3)(a^3-2^3)$

$= (a+2)(a^2-2a+4)\times(a-2)(a^2+2a+4)$

$= (a+2)(a-2)(a^2+2a+4)(a^2-2a+4)$

Activity : Resolve into factors:

1. $2x^4+16x$ 2. $8-a^3+3a^2b-3ab^2+b^3$ 3. $(a+b)^3+(a-b)^3$

(h) Factors of the expression with fractional coefficients :

Factors of the expressions with fraction may be expressed in different ways.

For example, $a^3+\dfrac{1}{27}=a^3+\dfrac{1}{3^3}=\left(a+\dfrac{1}{3}\right)\left(a^2-\dfrac{a}{3}+\dfrac{1}{9}\right)$

Again, $a^3+\dfrac{1}{27}=\dfrac{1}{27}(27a^3+1)=\dfrac{1}{27}\{(3a)^3+(1)^3\}$

$$= \frac{1}{27}(3a+1)(9a^2-3a+1)$$

Here, in the second solution, the factors involving the variables are with integral coefficients. This result can be expressed as the first solution :

$$\frac{1}{27}(3a+1)(9a^2-3a+1)$$
$$= \frac{1}{3}(3a+1)\times\frac{1}{9}(9a^2-3a+1)$$
$$= \left(a+\frac{1}{3}\right)\left(a^2-\frac{a}{3}+\frac{1}{9}\right)$$

Example 13. Resolve into factors : $x^3+6x^2y+11xy^2+6y^3$.

Solution : $x^3+6x^2y+11xy^2+6y^3$

$$= \{x^3+3\cdot x^2\cdot 2y+3\cdot x(2y)^2+(2y)^3\}-xy^2-2y^3$$
$$= (x+2y)^3-y^2(x+2y)$$
$$= (x+2y)\{(x+2y)^2-y^2\}$$
$$= (x+2y)(x+2y+y)(x+2y-y)$$
$$= (x+2y)(x+3y)(x+y)$$
$$= (x+y)(x+2y)(x+3y)$$

Activity : Resolve into factors:

1. $\frac{1}{2}x^2+\frac{7}{6}x+\frac{1}{3}$ 2. $a^3+\frac{1}{8}$ 3. $16x^2-25y^2-8xz+10yz$

Exercise 3·3

Resolve into factors **(1 to 43) :**

1. $a^2+ab+ac+bc$
2. $ab+a-b-1$
3. $(x-y)(x+y)+(x-y)(y+z)+(x-y)(z+x)$
4. $ab(x-y)-bc(x-y)$
5. $9x^2+24x+16$
6. a^4-27a^2+1
7. $x^4-6x^2y^2+y^4$
8. $(a^2-b^2)(x^2-y^2)+4abxy$
9. $4a^2-12ab+9b^2-4c^2$
10. $9x^4-45a^2x^2+36a^4$
11. $a^2+6a+8-y^2+2y$
12. $16x^2-25y^2-8xz+10yz$
13. $2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$
14. $x^2+13x+36$
15. x^4+x^2-20
16. $a^2-30a+216$

17. $x^6y^6 - x^3y^3 - 6$
18. $a^8 - a^4 - 2$
19. $a^2b^2 - 8ab - 105$
20. $x^2 - 37a - 650$
21. $4x^4 - 25x^2 + 36$
22. $12x^2 - 38x + 20$
23. $9x^2y^2 - 5xy^2 - 14y^2$
24. $4x^4 - 27x^2 - 81$
25. $ax^2 + (a^2+1)x + a$
26. $3(a^2+2a)^2 - 22(a^2+2a) + 40$
27. $14(x+z)^2 - 29(x+z)(x+1) - 15(x+1)^2$
28. $(4a-3b)^2 - 2(4a-3b)(a+2b) - 35(a+2b)^2$
29. $(a-1)x^2 + a^2xy + (a+1)y^2$
30. $24x^4 - 3x$
31. $(a^2+b^2)^3 + 8a^3b^3$
32. $x^3 + 3x^2 + 3x + 2$
33. $a^3 - 6a^2 + 12a - 9$
34. $a^3 - 9b^3 + (a+b)^3$
35. $8x^3 + 12x^2 + 6x - 63$
36. $8a^3 + \frac{b^3}{27}$
37. $a^3 - \frac{1}{8}$
38. $\frac{a^6}{27} - b^6$
39. $4a^2 + \frac{1}{4a^2} - 2 + 4a - \frac{1}{a}$
40. $(3a+1)^3 - (2a=3)^3$
41. $(x+5)(x-9) - 15$
42. $(x+2)(x+3)(x+4)(x+5) - 48$
43. $(x-1)(x-3)(x-5)(x-7) - 64$
44. Show that, $x^3 + 9x^2 + 26x + 24 = (x+2)(x+3)(x+4)$
45. Show that, $(x+1)(x+2)(3x-1)(3x-4) = (3x^2+2x-1)(3x^2+2x-8)$

3·5 Remainder Theorem

We observe the following example :

If $6x^2 - 7x + 5$ is divided by $x - 1$, the what is quotient and remainder?

Dividing $6x^2 - 7x + 5$ by $x - 1$ in common way, we get,

$$
\begin{array}{l}
x-1\ )\ 6x^2 - 7x + 5\ (\ 6x - 1 \\
\qquad\quad \underline{6x^2 \underset{+}{-} 6x} \\
\qquad\qquad\ -x + 5 \\
\qquad\qquad\ \underline{\underset{+}{-}x \underset{-}{+} 1} \\
\qquad\qquad\qquad\ 4
\end{array}
$$

Here, $x - 1$ is divisor, $6x^2 - 7x + 5$ is dividend, $6x - 1$ is quotient and 4 is remainder.

We know, dividend = divisor × quotient + remainder

Now, if we indicate the dividend by $f(x)$, the quotient by $h(x)$, the remainder by r and the divisor by $(x-a)$, from the above formula, we get,

$f(x)=(x-a)\cdot h(x)+r$ .... (i) this formula is true to all values of a.

Substituting $x=a$ in both sides of (i), we get,

$f(a)=(a-a)\cdot h(a)+r=0\cdot h(a)+r=r$

Hence, $r=f(a)$.

Therefore, if $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$. This formula is known as remainder theorem. That is, the remainder theorem gives the remainder when a polynomial $f(x)$ of positive degree is divided by $(x-a)$ without performing actual division. The degree of the divisor polynomial $(x-a)$ is 1. If the divisor is a factor of the dividend, the remainder will be zero and if it is not zero, the remainder will be a number other than zero.

Proposition : If the degree of $f(x)$ is positive and $a\neq 0$, $f(x)$ is divided by $(ax+b)$, remainder is $f\left(-\frac{b}{a}\right)$.

Proof : Degree of the divisor $ax+b$, $(a\neq 0)$ is 1.

Hence, we can write,

$$f(x)=(ax+b)\cdot h(x)+r=a\left(x+\frac{b}{a}\right)\cdot h(x)+r$$

$\therefore\ f(x)=\left(x+\frac{b}{a}\right)\cdot a\cdot h(x)+r$

Observe that, if $f(x)$ is divided by $\left(x+\frac{b}{a}\right)$, quotient is $a\cdot h(x)$ and remainder is r.

Here, divisor = $x-\left(-\frac{b}{a}\right)$

Hence, according to remainder theorem, $r=f\left(-\frac{b}{a}\right)$

Therefore, if $f(x)$ is divided by $(ax+b)$, remainder is $f\left(-\frac{b}{a}\right)$.

Corollary : $(x-a)$ will be a factor of $f(x)$, if and only if $f(a)=0$.

Proof : Let, $f(a)=0$

Therefore, according to remainder theorem, if $f(x)$ is divided by $(x-a)$, the remainder will be zero. That is, $(x-a)$ will be a factor of $f(x)$.

Conversely, let, $(x-a)$ is a factor of $f(x)$.

Therefore, $f(x)=(x-a)\cdot h(x)$, where $h(x)$ is a polynomial.

Putting $x=a$ in both sides, we get,

$f(a) = (a - a) \cdot h(a) = 0$

$\therefore\ f(a) = 0$.

Hence, any polynomial $f(x)$ will be divisible by $(x - a)$, if and only if $f(a) = 0$. This formula is known as factorisation theorem or factor theorem.

Corollary : If $a \neq 0$, the polynomial $ax + b$ will be a factor of any polynomial $f(x)$, if and only if $f\left(-\dfrac{b}{a}\right) = 0$.

Proof : $a \neq 0, ax + b = a\left(x + \dfrac{b}{a}\right)$ will be a factor of $f(x)$, if and only if $\left(x + \dfrac{b}{a}\right) = x - \left(-\dfrac{b}{a}\right)$ is a factor of $f(x)$, i.e. if and only if $f\left(-\dfrac{b}{a}\right) = 0$. This method of determining the factors of polynomial with the help of the remainder theorem is also called the Vanishing method.

Example 1. Resolve into factors : $x^3 - x - 6$.

Solution : Here, $f(x) = x^3 - x - 6$ is a polynomial. The factors of the constant -6 are $\pm 1, \pm 2, \pm 3$ and ± 6.

Putting, $x = 1, -1$, we see that the value of $f(x)$ is not zero.

But putting $x = 2$, we see that the value of $f(x)$ is zero.

i.e., $f(2) = 2^3 - 2 - 6 = 8 - 2 - 6 = 0$

Hence, $x - 2$ is a factor of $f(x)$

$$\begin{aligned}
\therefore\ f(x) &= x^3 - x - 6 \\
&= x^3 - 2x^2 + 2x^2 - 4x + 3x - 6 \\
&= x^2(x-2) + 2x(x-2) + 3(x-2) \\
&= (x-2)(x^2 + 2x + 3)
\end{aligned}$$

Example 2. Resolve into factors : $x^3 - 3xy^2 + 2y^3$.

Solution: Here, consider x a variable and y a constant.

We consider the given expression a polynomial of x.

Let, $f(x) = x^3 - 3xy^2 + 2y^3$

Then, $f(y) = y^3 - 3y \cdot y^2 + 2y^3 = 3y^3 - 3y^3 = 0$

$\therefore\ (x - y)$ is a factor of $f(x)$.

Now, $x^3 - 3xy^2 + 2y^3$	Again let, $g(x) = x^2 + xy - 2y^2$
$= x^3 - x^2y + x^2y - xy^2 - 2xy^2 + 2y^3$	$\therefore\ g(y) = y^2 + y^2 - 2y^2 = 0$

$= x^2(x-y)+xy(x-y)-2y^2(x-y)$

$= (x-y)(x^2+xy-2y^2)$

$= (x-y)(x^2+2xy-xy-2y^2)$

$= (x-y)\{x(x+2y)-y(x+2y)\}$

$= (x-y)(x+2y)(x-y)$

$= (x-y)^2(x+2y)$

$\therefore (x-y)$ is a factor of $g(x)$

$\therefore\ x^2+xy-2y^2$

$= x^2-xy+2xy-2y^2$

$= x(x-y)+2y(x-y)$

$= (x-y)(x+2y)$

$\therefore\ x^3-3xy^2+2y^3=(x-y)^2(x+2y)$

Example 3. Resolve into factors : $54x^4+27x^3a-16x-8a$.

Solution : Let, $f(x)=54x^4+27x^3a-16x-8a$

then, $f\left(-\frac{1}{2}a\right)=54\left(-\frac{1}{2}a\right)^4+27a\left(-\frac{1}{2}a\right)^3-16\left(-\frac{1}{2}a\right)-8a$

$$=\frac{27}{8}a^4-\frac{27}{8}a^4+8a-8a=0$$

$\therefore\ x-\left(-\frac{1}{2}a\right)=x+\frac{a}{2}$ i.e., $2x+a$ is a factor of $f(x)$.

Now, $54x^4+27x^3a-16x-8a=27x^3(2x+a)-8(2x+a)=(2x+a)(27x^3-8)$

$=(2x+a)\{(3x)^3-(2)^3\}=(2x+a)(3x-2)(9x^2+6x+4)$

Activity : Resolve into factors :

1. $x^3-21x-20$ 2. $2x^3-3x^2+3x-1$ 3. $x^3+6x^2+11x+6$

Exercise 3·4

Resolve into factors :

1.	$6x^2-7x+1$	2.	$3a^3+2a+5$
3.	$x^3-7xy^2-6y^3$	4.	x^2-5x-6
5.	$2x^2-x-3$	6.	$3x^2-7x-6$
7.	x^3+2x^2-5x-6	8.	x^3+4x^2+x-6
9.	$a^3+3a+36$	10.	a^4-4a+3
11.	$a^3-a^2-10a-8$	12.	x^3-3x^2+4x-4
13.	$a^3-7a^2b+7ab^2-b^3$	14.	x^3-x-24
15.	$x^3+6x^2y+11xy^2+6y^3$	16.	$2x^4-3x^3-3x-2$
17.	$4x^4+12x^3+7x^2-3x-2$	18.	$x^6-x^5+x^4-x^3+x^2-x$
19.	$4x^3-5x^2+5x-1$	20.	$18x^3+15x^2-x-2$

3·6 Forming and applying algebraic formulae in solving real life problems

In our daily business we face the realistic problems in different time and in different ways. These problems are described linguistically. In this section, we shall discuss the formation of algebraic formulae and their applications in solving different problems of real surroundings which are described linguistically. As a result of this discussion the students on the one hand, will get the conception about the application of mathematics in real surroundings, on the other hand, they will be eager to learn mathematics for their understanding of the involvement of mathematics with their surroundings.

Methods of solving the problems :

(a) At first the problem will have to be observed carefully and to read attentively and then to identify which are unknown and which are to be determined.

(b) One of the unknown quantities is to be denoted with any variable (say x). Then realising the problem well, express other unknown quantities in terms of the same variable (x).

(c) The problem will have to be splitted into small parts and express them by algebraic expressions.

(d) Using the given conditions, the small parts together are to be expressed by an equation.

(e) The value of the unknown quantity x is to be found by solving the equation.

Different formulae are used in solving the problems based on real life. The formulae are mentioned below :

(1) Related to Payable or Attainable :

Payable or attainable, $A =$ Tk. qn

where, q = amount of money payable or attainable per person,

n = number of person.

(2) Related to Time and Work :

If some persons perform a work,

Amount of work done, $W = qnx$

where, q = portion of a work performed by every one in unit of time.

n = number of performing of work

x = total time of doing work

$W =$ portion of a work done by n persons in time x.

(3) Related to Time and Distance :

Distance at a definite time, $d = vt$.

where, v = speed per hour
t = total time.

(4) Related to pipe and water tank :

Amount of water in a tank at a definite time, $Q(t) = Q_o \pm qt$

where, Q_o = amount of stored water in a tank at the time of opening the pipe
q = amount of water flowing in or flowing out by the pipe in a unit time.
t = time taken.
$Q(t)$ = amount of water in the tank in time t (+sign, at the time of flowing of water in and −sign, at the time of flowing of water out are to be used).

(5) Related to percentage :

$p = br$,

where, b = total quantity
r = (rate of) percentage by fraction $= \frac{s}{100} = s\%$
p = (rate of) percentage by parts = $s\%$ of b

(6) Related to profit and loss :

$S = C(I \pm r)$;

in case of profit, $S = C(I + r)$

in case of loss, $S = C(I - r)$

where, S (Tk.) = selling price
C (Tk.) = cost price
I = profit
r = rate of profit or loss

(7) Related to investment and profit :

In the case of simple profit,

$I = Pnr$ (taka)

$A = P + I = P + Pnr = P(1 + nr)$ (taka)

In the case of compound profit,

$A = P(1 + r)^n$

where, I = profit after time n
n = specific time
P = principal

r = profit of unit principal at unit time

A = principal with profit after time n.

Example 1. For a function of Annual Sports, members of an association made a budget of Tk. 45000 and divided that every member would subscribe equally. But 5 members refused to subscribe. As a result, amount of subscription of each member increased by Tk. 15 per head. How many members were in the association ?

Solution : Let the number of members of the association be x and amount of subscription per head be Tk. q Then total amount of subscription A = Tk. qx.

Actually numbers of members were $(x-5)$ and amount of subscription per head became Tk. $(q+15)$.

Then, total amount of subscription = Tk. $(x-5)(q+15)$

By the question, $qx=(x-5)(q+15)..........(i)$

and $qx=45{,}000..........(ii)$

From equation (i), we get, $qx=(x-5)(q+15)$

or, $qx=qx-5q+15x-75$

or, $5q=15x-75=5(3x-15)$

$\therefore\ q=3x-15...........(iii)$

Putting the value of q in equation (ii),

$(3x-15)\times x=45000$

or, $3x^2-15x=45000$

or, $x^2-5x=15000$ [dividing both sides by 3]

or, $x^2-5x-15000=0$

or, $x^2-125x+120x-15000=0$

or, $x(x-125)+120(x-125)=0$

or, $(x-125)(x+120)=0$

$\therefore\ x-125=0$ or, $x+120=0$

If $x-125=0$, $x=125$

Again, if $(x+120)=0$, $x=-120$

Since the number of members i.e,, x cannot be negative, $x\neq-120$.

$\therefore\ x=125$

Hence, number of members of the association is 125.

Example 2. Rafiq can do a work in 10 days and shafiq can do that work in 15 days. In how many days do they together finish the work ?

Solution : Let, Rafiq and Shafiq together can finish the wok in d days.

Let us make the following table :

Name	**Number of days for doing the work**	**Part of the work done in 1 day**	**Work done in d days**
Rafiq	10	$\frac{1}{10}$	$\frac{d}{10}$
Shafiq	15	$\frac{1}{15}$	$\frac{d}{15}$

By the questions, $\frac{d}{10}+\frac{d}{15}=1$

or, $d\left(\frac{1}{10}+\frac{1}{15}\right)=1$

or, $d\left(\frac{3+2}{30}\right)=1$

or, $\frac{5d}{30}=1$

or, $d=\frac{30}{5}=6$

$\therefore$ They together can finish the work in 6 days.

Example 3. A boatman can go x km in time t_1 hour against the current. To cover that distance along the current he takes t_2 hour. How much is the speed of the boat and the current.

Solution : Let the speed of the boat in still water be u km per hour and that of the current be v km per hour.

Then, along the current, the effective speed of boat is $(u+v)$ km per hour and against the current, the effective speed of boat is $(u-v)$ km per hour.

According to the question,

$$u+v=\frac{x}{t_2}.......(i) \quad [\because \text{speed}=\frac{\text{distance traversed}}{\text{time}}]$$

and $u-v=\frac{x}{t_1}.......(ii)$

Adding equations (i) and (ii) we get,

$$2u=\frac{x}{t_1}+\frac{x}{t_2}=x\left(\frac{1}{t_1}+\frac{1}{t_2}\right)$$

or, $u = \frac{x}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$

Subtracting equation (ii) from equation (i) we get,

$$2v = x\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$$

$$\text{or, } v = \frac{x}{2}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$$

Hence, speed of current is $\frac{x}{2}\left(\frac{1}{t_2} - \frac{1}{t_1}\right)$ km per hour

and speed of boat is $\frac{x}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$ km per hour.

Example 4. A pipe can fill up an empty tank in 12 minutes. Another pipe flows out 14 litre of water per minute. If the two pipes are opened together and the empty tank is filled up in 96 minutes, how much water does the tank contain ?

Solution : Let x litre of water flows in per minute by the first pipe and the tank can contain y litre of water.

According to the question, the tank is filled up by first pipe in 12 minutes,

$$\therefore \quad y = 12x \,.......(i)$$

Again, the empty tank is filled up by the two pipes together in 96 minutes.

$$\therefore \quad y = 96x - 96 \times 14 \,.........(ii)$$

From equation (i), we get, $x = \frac{y}{12}$

putting the value of x in equation (ii), we get,

$$y = 96 \times \frac{y}{12} - 96 \times 14$$

or, $y = 8y - 96 \times 14$

or, $7y = 96 \times 14$

or, $y = \frac{96 \times 14}{7} = 192$

Hence, total 192 litre of water is contained in the tank.

Activity :

1. For a picnic, a bus was hired at Tk. 2400 and it was decided that every passenger would have to give equal fare. But due to the absence of 10 passengers, fare per head was increased by Tk. 8. How many passengers did go by the bus and how much money did each of the passengers give as fare?
2. A and B together can do a work in p days. A alone can do that work in q days. In how many days can B alone do the work ?
3. A person rowing against the current can go 2 km per hour. If the speed of the current is 3 km per hour, how much time will he take to cover 32 km, rowing along the current ?

Example 5. Price of a book is Tk. 24.00. This price is 80% of the actual price. The Government subsidize the due price. How much money does the Govt. subsidize for each book ?

Solution : Market price = 80% of actual price

We know, $p = br$

Here, $p = \text{Tk. } 24$ and $r = 80\% = \dfrac{80}{100}$

$$\therefore\ 24 = b \times \frac{80}{100}$$

or, $b = \dfrac{\overset{6}{\cancel{24}} \times \overset{5}{\cancel{100}}}{\underset{\underset{1}{\cancel{4}}}{\cancel{80}}}$ $\quad \therefore\ b = 30$

Hence, the actual price of the book is Tk. 30.

$\therefore$ amount of subsidized money = Tk. $(30 - 24)$

= Tk. 6.

$\therefore$ subsidized money for each book is Tk. 6.

Example 6. The loss is $r\%$ when n oranges are sold per taka. How many oranges are to be sold per taka to make a profit of $s\%$?

Solution : If the cost price is Tk. 100, the selling price at the loss of $r\%$ is Tk. $(100 - r)$.

If selling price is Tk. $(100 - r)$, cost price is Tk. 100

$\therefore$,, ,, ,, ,, Tk. 1 ,, ,, ,, Tk. $\dfrac{100}{100 - r}$

Again, if cost price is Tk. 100, selling price at the profit of $s\%$ is Tk. $(100+s)$

$\therefore$,, ,, ,, ,, Tk. 1 ,, ,, ,, ,, ,, ,, ,, ,, Tk. $\dfrac{100+s}{100}$

$\therefore$,, ,, ,, ,, Tk. $\dfrac{100}{100-r}$,, ,, ,, ,, ,, ,, ,, Tk. $\left(\dfrac{100+s}{100}\times\dfrac{100}{100-r}\right)$

$= \text{Tk. } \dfrac{100+s}{100-r}$

$\therefore$ in Tk. $\dfrac{100+s}{100-r}$, number of oranges is to be sold = n

$\therefore$ in Tk. 1, ,, ,, ,, ,, ,, ,, ,, $n\times\left(\dfrac{100-r}{100+s}\right)$

Hence, $\dfrac{n(100-r)}{100+s}$ oranges are to be sold per taka.

Example 7. What is the profit of Tk. 650 in 6 years at the rate of profit Tk. 7 percent per annum ?

Solution : We know , $I = Pnr$.

Here, $P =$ Tk. 650, $n = 6$, $s = 7$

$\therefore \quad r = \dfrac{s}{100} = \dfrac{7}{100}$

$\therefore \quad I = 650\times6\times\dfrac{7}{100} = 273$

Hence, profit is Tk. 273.

Example 8. Find the compound principal and compound profit of Tk. 15000 in 3 years at the profit of 6 percent per annum.

Solution : We know, $C = P(1+r)^n$, where C is the profit principal in the case of compound profit.

Gven, $P = \text{Tk.}15000$, $r = 6\% = \dfrac{6}{100}$, $n = 3$ years

$\therefore \; C = 15000\left(1+\dfrac{6}{100}\right)^3 = 15000\left(1+\dfrac{3}{50}\right)^3$

$= 15000\left(\dfrac{53}{50}\right)^3$

$$= \frac{\overset{3}{\cancel{15}} \times 53 \times 53 \times 53}{\underset{25}{\cancel{125}}} = \frac{3 \times 148877}{25}$$

$$= \frac{446631}{25} = 17865 \cdot 24$$

$\therefore$ compound principal is Tk. $17865 \cdot 24$

$\therefore$ compound profit = Tk. $(17865 \cdot 24 - 15000)$

= Tk. $2865 \cdot 24$

Activity :

1. The loss is $n\%$ when 10 lemons are sold per taka. How many lemons are to be sold per taka to make the profit of $z\%$?
2. What will be the profit principal of Tk. 750 in 4 years at the rate of simple profit $6\frac{1}{2}$ percent per annum ?
3. Find the compound principal of Tk. 2000 in 3 years at the rate of compound profit of Tk. 4 percent per annum.

Exercise 3·5

1. Which one of the following is the factorized form of $x^2 - 7x + 6$?

   (a) $(x-2)(x-3)$ (b) $(x-1)(x+8)$

   (c) $(x-1)(x-6)$ (d) $(x+1)(x+6)$

2. If $f(x) = x^2 - 4x + 4$, which one of the following is the value of $f(2)$?

   (a) 4 (b) 2

   (c) 1 (d) 0

3. If $x + y = x - y$, which one of the following is the value of y ?

   (a) -1 (b) 0

   (c) 1 (d) 2

4. Which one of the following is the lowest form of $\frac{x^2 + 3x^3}{x + 3x^2}$?

   (a) x^2 (b) x

   (c) 1 (d) 0

5. Which one of the following is the lowest form of $\frac{1 - x^2}{1 - x}$?

   (a) 1 (b) x

6. Which one of the following is the value of $\frac{1}{2}\{(a+b)^2-(a-b)^2\}$?

(a) $2(a^2+b^2)$ (b) a^2+b^2

(c) $2ab$ (d) $4ab$

7. If $x+\frac{2}{x}=3$, what is the value of $x^3+\frac{8}{x^3}$?

(a) 1 (b) 8 (c) 9 (d) 16

8. Which one of the following is the factorized form of p^4+p^2+1 ?

(a) $(p^2-p+1)(p^2+p-1)$ (b) $(p^2-p-1)(p^2+p+1)$

(c) $(p^2+p+1)(p^2+p+1)$ (d) $(p^2+p+1)(p^2-p+1)$

9. What are the factors of x^2-5x+4?

(a) $(x-1),(x-4)$ (b) $(x+1),(x-4)$

(c) $(x+2),(x-2)$ (d) $(x-5),(x-1)$

10. What is the value of $(x-7)(x-5)$?

(a) $x^2+12x+35$ (b) $x^2+12x-35$

(c) $x^2-12x+35$ (d) $x^2-12x-35$

11. What is the value of $\frac{2\cdot9\times2\cdot9-1\cdot1\times1\cdot1}{2\cdot9-1\cdot1}$?

(a) $1\cdot8$ (b) $1\cdot9$

(c) 2 (d) 4

12. If $x=2-\sqrt{3}$, what is the value of x^2 ?

(a) 1 (b) $7-4\sqrt{3}$

(c) $2+\sqrt{3}$ (d) $\frac{1}{2-\sqrt{3}}$

13. If $f(x)=x^2-5x+6$ and $f(x)=0$, x = what ?

(a) 2, 3 (b) $-5, 1$

(c) $-2, 3$ (d) 1, 5

14.

	x	$+6$
x	x^2	$+6x$
-5	$-5x$	-30

Which one of the following is the total area of the table above ?

(a) $x^2 - 5x + 30$ (b) $x^2 + x - 30$

(c) $x^2 + 6x - 30$ (d) $x^2 - x + 30$

15. A can do a work in x days, B can do that work in $3x$ days. In the same time, how many times does A of B work ?

(a) 2 times (b) $2\frac{1}{2}$ times

(c) 3 times (d) 4 times

16. If $a + b = -c$, $a^2 + 2ab + b^2$ is expressed in terms of c, which one of the following will be ?

(a) $-c^2$ (b) c^2 (c) bc (d) ca

17. If $x + y = 3, xy = 2$, what is the value of $x^3 + y^3$?

(a) 9 (b) 18

(c) 19 (d) 27

18. Which one is the factorized form of $8x^3 + 27y^3$?

(a) $(2x - 3y)(4x^2 + 6xy + 9y^2)$ (b) $(2x + 3y)(4x^2 - 6xy + 9y^2)$

(c) $(2x - 3y)(4x^2 - 9y^2)$ (d) $(2x + 3y)(4x^2 + 9y^2)$

19. What is to be added to $9x^2 + 16y^2$, so that their sum will be a perfect square ?

(a) $6xy$ (b) $12xy$

(c) $24xy$ (d) $144xy$

20. If $x - y = 4$, which one of the following statements is correct ?

(a) $x^3 - y^3 - 4xy = 64$ (b) $x^3 - y^3 - 12xy = 12$

(c) $x^3 - y^3 - 3xy = 64$ (d) $x^3 - y^3 - 12xy = 64$

21. If $x^4 - x^2 + 1 = 0$,

(1) $x^2 + \frac{1}{x^2}$ = what ?

(a) 4 (b) 2 (c) 1 (d) 0

(2) What is the value of $\left(x + \frac{1}{x}\right)^2$?

(a) 4 (b) 3

(c) 2 (d) 1

(3) $x^3 + \frac{1}{x^3} =$ what ?

(a) 3 (b) 2

(c) 1 (d) 0

22. A can do a work in p days and B can do it in $2p$ days. They started to do the work together and after some days A left the work unfinished. B completed the rest of the work in r days. In how many days was the work finished ?
23. 50 persons can do a work in 12 days by working 8 hours a day. Working how many hours per day can 60 persons finish the work in 16 days. ?
24. Mita can do a work in x days and Rita can do that work in y days. In how many days will they together complete the work ?
25. A bus was hired at Tk. 57000 to go for a picnic under the condition that every passenger would bare equal fare. But due to the absence of 5 passengers, the fare was increased by Tk. 3 per head. How many passengers availed the bus ?
26. A boatman can go d km in p hours against the current. He takes q hours to cover that distance along the current. What is the speed of the current and the boat ?
27. A boatman plying by oar goes 15 km and returns from there in 4 hours. He goes 5 km at a period of time along the current and goes 3 km at the same period of time against the current. Find the speed of the oar and current.
28. Two pipes are connected with a tank. The empty tank is filled up in t_1 minutes by the first pipe and it becomes empty in t_2 minutes by the second pipe. If the two pipes are opened together, in how much time will the tank be filled up ? (here $t_1 > t_2$)
29. A tank is filled up in 12 minutes by a pipe. Another pipe flows out 15 litre of water in 1 minute. When the tank remains empty, the two pipes are opened togethr and the tank is filled up in 48 minutes. How much water does the tank contain ?
30. If a pen is sold at Tk. 11, there is a profit of 10%. What was the cost price of the pen ?
31. Due to the sale of a notebook at Tk. 36, there was a loss. If the notebook would be sold at Tk. 72, there would be profit amounting twice the loss. What was the cost price of the notebook ?
32. Divide Tk. 260 among A, B and C in such a way that 2 times the share of A, 3 times the share of B and 4 times the share of C are equal to one another.
33. Due to the selling of a commodity at the loss of $x\%$ such price is obtained that due to the selling at the profit of $3x\%$ Tk. $18x$ more is obtained. What was the cost price of the commodity ?

34. If the simple profit of Tk. 300 in 4 years and the simple profit of Tk. 400 in 5 years together are Tk. 148, what is the percentage of profit ?
35. If the difference of simple profit and compound profit of some principal in 2 years is Tk. 1 at the rate of profit 4%, what is the principal ?
36. Some principal beomes Tk. 460 with simple profit in 3 years and Tk. 600 with simple profit in 5 years. What is the rate of profit ?
37. How much money will become Tk. 985 as profit principal in 13 years at the rate of simple profit 5% per annum ?
38. How much money will become Tk. 1248 as profit principal in 12 years at the rate of profit 5% per annum ?
39. Find the difference of simple profit and compound profit of Tk. 8000 in 3 years at the rate of profit 5%.
40. The Value Added Tax (VAT) of sweets is $x\%$. If a trader sells sweets at Tk. P including VAT, how much VAT is he to pay ? If $x = 15,\ P = 2300$, what is the amount of VAT ?
41. Sum of a number and its multiplicative inverse is 3.

    (a) Taking the number as the variable x, express the above information by an equation.

    (b) Find the value of $x^3 - \frac{1}{x^3}$.

    (c) Prove that, $x^5 - \frac{1}{x^5} = 123$.
42. Each of the members of an association decided to subscribe 100 times the number of members. But 7 members did not subscribe. As a result, amount of subscription for each member was increased by Tk. 500 than the previous.

    (a) If the number of members is x and total amount of subscription is Tk. A, find the relation between them.

    (b) Find the number of members of the association and total amount of subscription.

    (c) $\frac{1}{4}$ of total amount of subscription at the rate of simple profit 5% and rest of the money at the rate of simple profit 4% were invested for 2 years. Find the total profit.